Contents

Preface **xi**

1 Math review **1**

1.1 Some sets 1

1.2 Pairs of reals 3

1.3 Exponentials and logs 4

1.4 Some handy functions 5

1.5 Summations 6

1.6 Strings 8

1.7 Variation in notation 9

2 Logic **10**

2.1 A bit about style 10

2.2 Propositions 11

2.3 Complex propositions 11

2.4 Implication 12

2.5 Converse, contrapositive, biconditional 14

2.6 Complex statements 15

2.7 Logical Equivalence 16

2.8 Some useful logical equivalences 18
2.9 Negating propositions 18
2.10 Predicates and Variables 19
2.11 Other quantifiers 20
2.12 Notation 22
2.13 Useful notation 22
2.14 Notation for 2D points 23
2.15 Negating statements with quantifiers 24
2.16 Binding and scope 25
2.17 Variations in Notation 26

3 Proofs **27**

3.1 Proving a universal statement 27
3.2 Another example of direct proof involving odd and even 29
3.3 Direct proof outline 30
3.4 Proving existential statements 31
3.5 Disproving a universal statement 31
3.6 Disproving an existential statement 32
3.7 Recap of proof methods 33
3.8 Direct proof: example with two variables 33
3.9 Another example with two variables 34
3.10 Proof by cases 35
3.11 Rephrasing claims 36
3.12 Proof by contrapositive 37
3.13 Another example of proof by contrapositive 38

4 Number Theory **39**

4.1 Factors and multiples 39
4.2 Direct proof with divisibility 40
4.3 Stay in the Set 41
4.4 Prime numbers 42
4.5 GCD and LCM 42
4.6 The division algorithm 43
4.7 Euclidean algorithm 44
4.8 Pseudocode 46
4.9 A recursive version of gcd 46
4.10 Congruence mod k 47
4.11 Proofs with congruence mod k 48
4.12 Equivalence classes 48
4.13 Wider perspective on equivalence 50
4.14 Variation in Terminology 51

5 Sets **52**

5.1 Sets 52
5.2 Things to be careful about 53
5.3 Cardinality, inclusion 54
5.4 Vacuous truth 55
5.5 Set operations 56
5.6 Set identities 58
5.7 Size of set union 58
5.8 Product rule 59
5.9 Combining these basic rules 60

5.10 Proving facts about set inclusion 61
5.11 An abstract example 63
5.12 An example with products 64
5.13 A proof using sets and contrapositive 65
5.14 Variation in notation 66

6 Relations **67**

6.1 Relations 67
6.2 Properties of relations: reflexive 69
6.3 Symmetric and antisymmetric 70
6.4 Transitive 71
6.5 Types of relations 73
6.6 Proving that a relation is an equivalence relation 74
6.7 Proving antisymmetry 75

7 Functions and onto **77**

7.1 Functions 77
7.2 When are functions equal? 79
7.3 What isn't a function? 80
7.4 Images and Onto 81
7.5 Why are some functions not onto? 82
7.6 Negating onto 82
7.7 Nested quantifiers 83
7.8 Proving that a function is onto 85
7.9 A 2D example 86
7.10 Composing two functions 87

7.11 A proof involving composition 88
7.12 Variation in terminology 88

8 Functions and one-to-one **90**

8.1 One-to-one 90
8.2 Bijections 92
8.3 Pigeonhole Principle 92
8.4 Permutations 93
8.5 Further applications of permutations 94
8.6 Proving that a function is one-to-one 95
8.7 Composition and one-to-one 96
8.8 Strictly increasing functions are one-to-one 97
8.9 Making this proof more succinct 98
8.10 Variation in terminology 99

9 Graphs **100**

9.1 Graphs 100
9.2 Degrees 102
9.3 Complete graphs 103
9.4 Cycle graphs and wheels 103
9.5 Isomorphism 104
9.6 Subgraphs 106
9.7 Walks, paths, and cycles 107
9.8 Connectivity 108
9.9 Distances 109
9.10 Euler circuits 110

9.11 Bipartite graphs 111
9.12 Variation in terminology 113

10 2-way Bounding **114**
10.1 Marker Making 115
10.2 Pigeonhole point placement 116
10.3 Graph coloring 117
10.4 Why care about graph coloring? 119
10.5 Proving set equality 120
10.6 Variation in terminology 121

11 Induction **122**
11.1 Introduction to induction 122
11.2 An Example 123
11.3 Why is this legit? 124
11.4 Building an inductive proof 125
11.5 Another example 126
11.6 Some comments about style 127
11.7 A geometrical example 128
11.8 Graph coloring 129
11.9 Postage example 131
11.10Nim 133
11.11Prime factorization 134
11.12Variation in notation 135

12 Recursive Definition **137**
12.1 Recursive definitions 137

12.2 Finding closed forms . . . 138
12.3 Divide and conquer . . . 140
12.4 Hypercubes . . . 142
12.5 Proofs with recursive definitions . . . 143
12.6 Inductive definition and strong induction . . . 144
12.7 Variation in notation . . . 145

13 Trees **146**

13.1 Why trees? . . . 146
13.2 Defining trees . . . 149
13.3 m-ary trees . . . 150
13.4 Height vs number of nodes . . . 151
13.5 Context-free grammars . . . 151
13.6 Recursion trees . . . 157
13.7 Another recursion tree example . . . 159
13.8 Tree induction . . . 160
13.9 Heap example . . . 161
13.10 Proof using grammar trees . . . 163
13.11 Variation in terminology . . . 164

14 Big-O **166**

14.1 Running times of programs . . . 166
14.2 Asymptotic relationships . . . 167
14.3 Ordering primitive functions . . . 169
14.4 The dominant term method . . . 170
14.5 Big-O . . . 171

14.6 Applying the definition of big-O . . . 172
14.7 Proving a primitive function relationship . . . 173
14.8 Variation in notation . . . 174

15 Algorithms 176

15.1 Introduction . . . 176
15.2 Basic data structures . . . 176
15.3 Nested loops . . . 178
15.4 Merging two lists . . . 179
15.5 A reachability algorithm . . . 180
15.6 Binary search . . . 181
15.7 Mergesort . . . 183
15.8 Tower of Hanoi . . . 184
15.9 Multiplying big integers . . . 186

16 NP 189

16.1 Finding parse trees . . . 189
16.2 What is NP? . . . 190
16.3 Circuit SAT . . . 192
16.4 What is NP complete? . . . 194
16.5 Variation in notation . . . 195

17 Proof by Contradiction 196

17.1 The method . . . 196
17.2 $\sqrt{2}$ is irrational . . . 197
17.3 There are infinitely many prime numbers . . . 198
17.4 Lossless compression . . . 199

17.5 Philosophy 200

18 Collections of Sets **201**

18.1 Sets containing sets 201

18.2 Powersets and set-valued functions 203

18.3 Partitions 204

18.4 Combinations 206

18.5 Applying the combinations formula 207

18.6 Combinations with repetition 208

18.7 Identities for binomial coefficients 209

18.8 Binomial Theorem 210

18.9 Variation in notation 211

19 State Diagrams **212**

19.1 Introduction 212

19.2 Wolf-goat-cabbage puzzle 214

19.3 Phone lattices 215

19.4 Representing functions 217

19.5 Transition functions 217

19.6 Shared states 218

19.7 Counting states 221

19.8 Variation in notation 222

20 Countability **223**

20.1 The rationals and the reals 223

20.2 Completeness 224

20.3 Cardinality 224

20.4 Cantor Schroeder Bernstein Theorem . . . 225
20.5 More countably infinite sets . . . 226
20.6 $\mathbb{P}(\mathbb{N})$ isn't countable . . . 227
20.7 More uncountability results . . . 228
20.8 Uncomputability . . . 229
20.9 Variation in notation . . . 231

21 Planar Graphs 232
21.1 Planar graphs . . . 232
21.2 Faces . . . 233
21.3 Trees . . . 234
21.4 Proof of Euler's formula . . . 235
21.5 Some corollaries of Euler's formula . . . 236
21.6 $K_{3,3}$ is not planar . . . 237
21.7 Kuratowski's Theorem . . . 238
21.8 Coloring planar graphs . . . 241
21.9 Application: Platonic solids . . . 242

A Jargon 245
A.1 Strange technical terms . . . 245
A.2 Odd uses of normal words . . . 246
A.3 Constructions . . . 248
A.4 Unexpectedly normal . . . 249

B Acknowledgements and Supplementary Readings 251

C Where did it go? 255