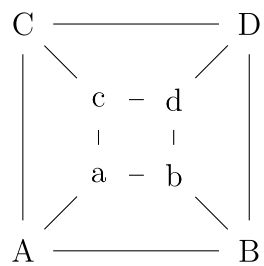
Chapter 21

Planar Graphs

This chapter covers special properties of planar graphs.

21.1 Planar graphs

A planar graph is a graph which can be drawn in the plane without any edges crossing. Some pictures of a planar graph might have crossing edges, but it's possible to redraw the picture to eliminate the crossings. For example, although the usual pictures of K_4 and Q_3 have crossing edges, it's easy to redraw them so that no edges cross. For example, a planar picture of Q_3 is shown below. However, if you fiddle around with drawings of $K_{3,3}$ or K_5, there doesn't seem to be any way to eliminate the crossings. We'll see how to prove that these two graphs aren't planar.

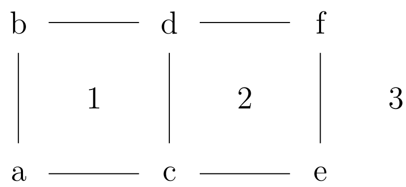


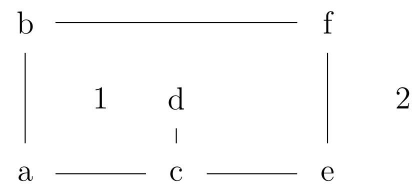
Why should we care? Planar graphs have some interesting mathematical properties, e.g. they can be colored with only 4 colors. Also, as we'll see later, we can use facts about planar graphs to show that there are only 5 Platonic solids.

There are also many practical applications with a graph structure in which crossing edges are a nuisance, including design problems for circuits, subways, utility lines. Two crossing connections normally means that the edges must be run at different heights. This isn't a big issue for electrical wires, but it creates extra expense for some types of lines e.g. burying one subway tunnel under another (and therefore deeper than you would ordinarily need). Circuits, in particular, are easier to manufacture if their connections live in fewer layers.

21.2 Faces

When a planar graph is drawn with no crossing edges, it divides the plane into a set of regions, called *faces*. By convention, we also count the unbounded area outside the whole graph as one face. The *boundary* of a face is the subgraph containing all the edges adjacent to that face and a *boundary walk* is a closed walk containing all of those edges. The *degree* of the face is the minimum length of a boundary walk. For example, in the figure below, the lefthand graph has three faces. The boundary of face 2 has edges df, fe, ec, cd, so this face has degree 4. The boundary of face 3 (the unbounded face) has edges bd, df, fe, ec, ca, ab, so face 3 has degree 6.





The righthand graph above has a spike edge sticking into the middle of face 1. The boundary of face 1 has edges bf, fe, ec, cd, ca, ab. However, any boundary walk must traverse the spike twice, e.g. one possible boundary walk is $bf, fe, ec, cd, cd, ca, ab$, in which cd is used twice. So the degree of face 1 in the righthand graph is 7. In such cases, it may help to think of walking along inside the face just next to the boundary, rather walking along the boundaries themselves. Notice that the boundary walk for such a face is not a cycle.

Suppose that we have a graph with e edges, v nodes, and f faces. We know that the Handshaking theorem holds, i.e. the sum of node degrees is $2e$. For planar graphs, we also have a *Handshaking theorem for faces*: the sum of the face degrees is $2e$. To see this, notice that a typical edge forms part of the boundary of two faces, one to each side of it. The exceptions are edges, such as those involved in a spike, that appear twice on the boundary of a single face.

Finally, for connected planar graphs, we have Euler's formula: $v-e+f = 2$. We'll prove that this formula works.[1]

21.3 Trees

Before we try to prove Euler's formula, let's look at one special type of planar graph: free trees. In graph theory, a **free tree** is any connected graph with no cycles. Free trees are somewhat like normal trees, but they don't have a designated root node and, therefore, they don't have a clear ancestor-descendent ordering to their notes.

A free tree doesn't divide the plane into multiple faces, because it doesn't

[1] You can easily generalize Euler's formula to handle graphs with more than one connected components.

contain any cycles. A free tree has only one face: the entire plane surrounding it. So Euler's theorem reduces to $v - e = 1$, i.e. $e = v - 1$. Let's prove that this is true, by induction.

> Proof by induction on the number of nodes in the graph.
>
> Base: If the graph contains no edges and only a single node, the formula is clearly true.
>
> Induction: Suppose the formula works for all free trees with up to n nodes. Let T be a free tree with $n + 1$ nodes. We need to show that T has n edges.
>
> Now, we find a node with degree 1 (only one edge going into it). To do this start at any node r and follow a walk in any direction, without repeating edges. Because T has no cycles, this walk can't return to any node it has already visited. So it must eventually hit a dead end: the node at the end must have degree 1. Call it p.
>
> Remove p and the edge coming into it, making a new free tree T' with n nodes. By the inductive hypothesis, T' has $n - 1$ edges. Since T has one more edge than T', T has n edges. Therefore our formula holds for T.

21.4 Proof of Euler's formula

We can now prove Euler's formula ($v - e + f = 2$) works in general, for any connected planar graph.

> Proof: by induction on the number of edges in the graph.
>
> Base: If $e = 0$, the graph consists of a single node with a single face surrounding it. So we have $1 - 0 + 1 = 2$ which is clearly right.
>
> Induction: Suppose the formula works for all graphs with no more than n edges. Let G be a graph with $n + 1$ edges.
>
> Case 1: G doesn't contain a cycle. So G is a free tree and we already know the formula works for free trees.

Case 2: G contains at least one cycle. Pick an edge p that's on a cycle. Remove p to create a new graph G'.

Since the cycle separates the plane into two faces, the faces to either side of p must be distinct. When we remove the edge p, we merge these two faces. So G' has one fewer faces than G.

Since G' has n edges, the formula works for G' by the induction hypothesis. That is $v' - e' + f' = 2$. But $v' = v$, $e' = e - 1$, and $f' = f - 1$. Substituting, we find that

$$v - (e - 1) + (f - 1) = 2$$

So

$$v - e + f = 2$$

21.5 Some corollaries of Euler's formula

Corollary 1 *Suppose G is a connected planar graph, with v nodes, e edges, and f faces, where $v \geq 3$. Then $e \leq 3v - 6$.*

Proof: The sum of the degrees of the faces is equal to twice the number of edges. But each face must have degree ≥ 3. So we have $3f \leq 2e$.

Euler's formula says that $v - e + f = 2$, so $f = e - v + 2$ and thus $3f = 3e - 3v + 6$. Combining this with $3f \leq 2e$, we get $3e - 3v + 6 \leq 2e$. So $e \leq 3v - 6$.

We can also use this formula to show that the graph K_5 isn't planar. K_5 has five nodes and 10 edges. This isn't consistent with the formula $e \leq 3v-6$. Unfortunately, this method won't help us with $K_{3,3}$, which isn't planar but does satisfy this equation.

We can also use this Corollary 1 to derive a useful fact about planar graphs:

Corollary 2 *If G is a connected planar graph, G has a node of degree less than six.*

> Proof: This is clearly true if G has one or two nodes.
>
> If G has at least three nodes, then suppose that the degree of each node was at least 6. By the handshaking theorem, $2e$ equals the sum of the degrees of the nodes, so we would have $2e \geq 6v$. But corollary 1 says that $e \leq 3v - 6$, so $2e \leq 6v - 12$. We can't have both $2e \geq 6v$ and $2e \leq 6v - 12$. So there must have been a node with degree less than six.

If our graph G isn't connected, the result still holds, because we can apply our proof to each connected component individually. So we have:

Corollary 3 *If G is a planar graph, G has a node of degree less than six.*

21.6 $K_{3,3}$ is not planar

When all the cycles in our graph have at least four nodes, we can get a tighter relationship between the numbers of nodes and edges.

Corollary 4 *Suppose G is a connected planar graph, with v nodes, e edges, and f faces, where $v \geq 3$. and if all cycles in G have length ≥ 4, then $e \leq 2v - 4$.*

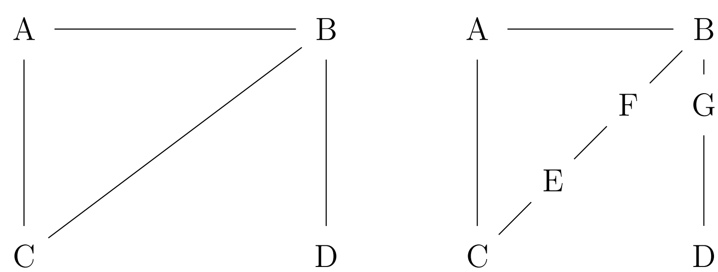
> Proof: The sum of the degrees of the faces is equal to twice the number of edges. But each face must have degree ≥ 4 because all cycles have length ≥ 4. So we have $4f \leq 2e$, so $2f \leq e$.
>
> Euler's formula says that $v - e + f = 2$, so $e - v + 2 = f$, so $2e - 2v + 4 = 2f$. Combining this with $2f \leq e$, we get that $2e - 2v + 4 \leq e$. So $e \leq 2v - 4$.

This result lets us show that $K_{3,3}$ isn't planar. All the cycles in $K_{3,3}$ have at least four nodes. But $K_{3,3}$ has 9 edges and 6 nodes, which isn't consistent with this formula. So $K_{3,3}$ can't be planar.

21.7 Kuratowski's Theorem

The two example non-planar graphs $K_{3,3}$ and K_5 weren't picked randomly. It turns out that any non-planar graph must contain a subgraph closely related to one of these two graphs. Specifically, we'll say that a graph G is a *subdivision* of another graph F if the two graphs are isomorphic or if the only difference between the graphs is that G divides up some of F's edges by adding extra degree 2 nodes in the middle of the edges.

For example, in the following picture, the righthand graph is a subdivision of the lefthand graph.

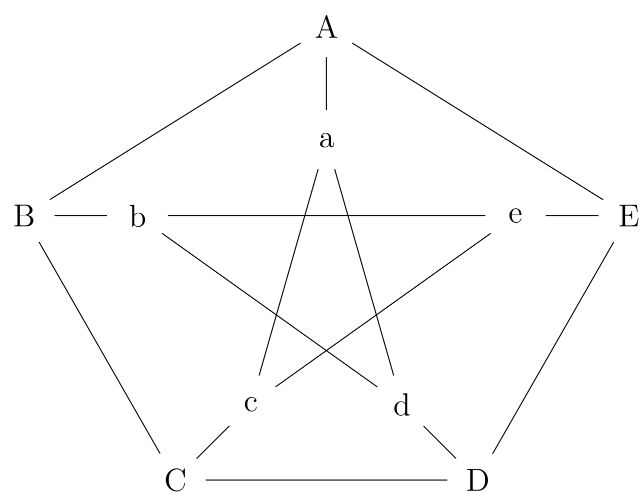


We can now state our theorem precisely.

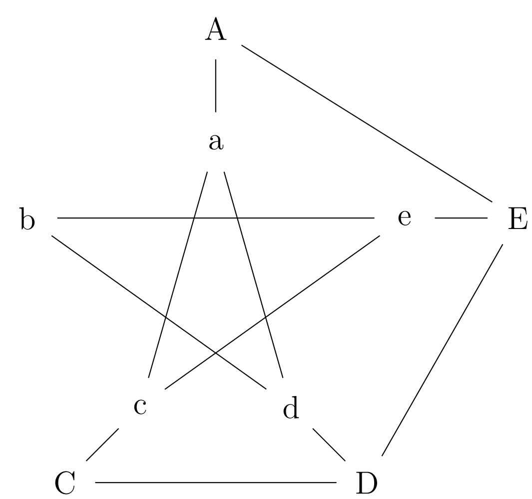
Claim 54 *Kuratowski's Theorem: A graph is nonplanar if and only if it contains a subgraph that is a subdivision of* $K_{3,3}$ *or* K_5*.*

This was proved in 1930 by Kazimierz Kuratowski, and the proof is apparently somewhat difficult. So we'll just see how to apply it.

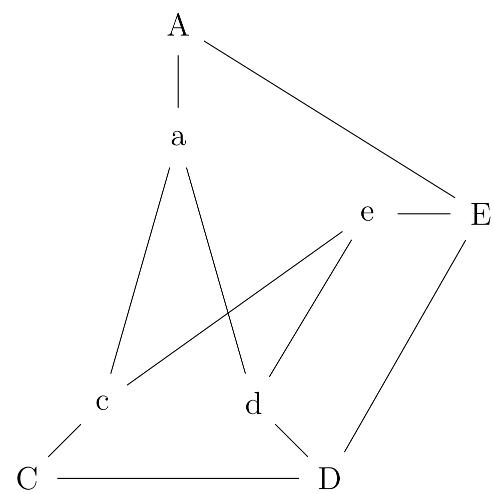
For example, here's a graph known as the Petersen graph (after a Danish mathematician named Julius Petersen).



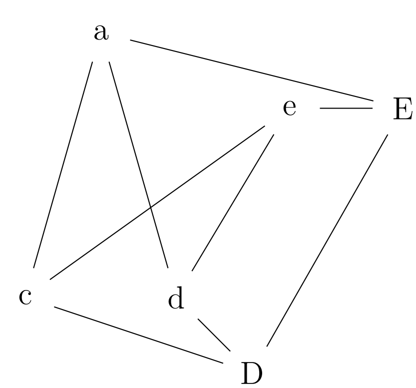
This isn't planar. The offending subgraph is the whole graph, except for the node B (and the edges that connect to B):



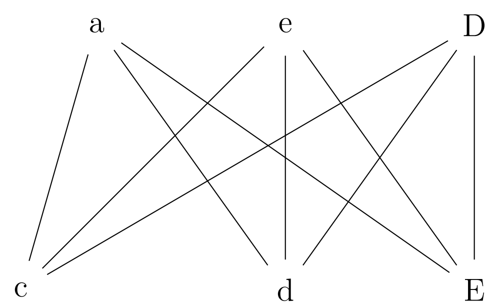
This subgraph is a subdivision of $K_{3,3}$. To see why, first notice that the node b is just subdividing the edge from d to e, so we can delete it. Or, formally, the previous graph is a subdivision of this graph:



In the same way, we can remove the nodes A and C, to eliminate unnecessary subdivisions:



Now deform the picture a bit and we see that we have $K_{3,3}$.



21.8 Coloring planar graphs

One application of planar graphs involves coloring maps of countries. Two countries sharing a border[2] must be given different colors. We can turn this into a graph coloring problem by assigning a graph node to each country. We then connect two nodes with an edge exactly when their regions share a border. This graph is called the *dual* of our original map. Because the maps are planar, these dual graphs are always planar.

Planar graphs can be colored much more easily than other graphs. For example, we can prove that they never require more than 6 colors:

> Proof: by induction on the number of nodes in G.
>
> Base: The planar graph with just one node has maximum degree 0 and can be colored with one color.
>
> Induction: Suppose that any planar graph with $< k$ nodes can be colored with 6 colors. Let G be a planar graph with k nodes.

[2]Two regions touching at a point are not considered to share a border.

> By Corollary 3, G has a node of degree less than 6. Let's pick such a node and call it v.
>
> Remove some node v (and its edges) from G to create a smaller graph G'. G' is a planar graph with $k-1$ nodes. So, by the inductive hypothesis, G' can be colored with 6 colors.
>
> Because v has less than 6 neighbors, its neighbors are only using 5 of the available colors. So there is a spare color to assign to v, finishing the coloring of G.

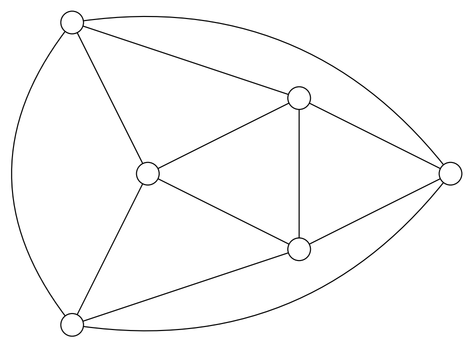
It's not hard, but a bit messy, to upgrade this proof to show that planar graphs require only five colors. Four colors is much harder. Way back in 1852, Francis Guthrie hypothesized that any planar graph could be colored with only four colors, but it took 124 years to prove that he was right. Alfred Kempe thought he had proved it in 1879 and it took 11 years for another mathematician to find an error in his proof.

The *Four Color Theorem* was finally proved by Kenneth Appel and Wolfgang Haken at UIUC in 1976. They reduced the problem mathematically, but were left with 1936 specific graphs that needed to be checked exhaustively, using a computer program. Not everyone was happy to have a computer involved in a mathematical proof, but the proof has come to be accepted as legitimate.

21.9 Application: Platonic solids

A fact dating back to the Greeks is that there are only five *Platonic solids*: cube, dodecahedron, tetrahedron, icosahedron, octahedron. These are convex polyhedra whose faces all have the same number of sides (k) and whose nodes all have the same number of edges going into them (d).

To turn a Platonic solid into a graph, imagine that it's made of a stretchy material. Make a small hole in one face. Put your fingers into that face and pull sideways, stretching that face really big and making the whole thing flat. For example, an octahedron (8 triangular sides) turns into the following graph. Notice that it still has eight faces, one for each face of the original solid, each with three sides.



Graphs of polyhedra are slightly special planar graphs. Polyhedra aren't allowed to have extra nodes partway along edges, so each node in the graph must have degree at least three. Also, since the faces must be flat and the edges straight, each face needs to be bounded by at least three edges. So, if G is the graph of a Platonic solid, all the nodes of G must have the same degree $d \geq 3$ and all faces must have the same degree $k \geq 3$.

Now, let's do some algebra to see why there are so few possibilities for the structure of such a graph.

By the handshaking theorem, the sum of the node degrees is twice the number of edges. So, since the degrees are equal to d, we have $dv = 2e$ and therefore

$$v = \frac{2e}{d}$$

By the handshaking theorem for faces, the sum of the face degrees is also twice the number of edges. That is $kf = 2e$. So

$$f = \frac{2e}{k}$$

Euler's formula says that $v - e + f = 2$, so $v + f = 2 + e > e$. Substituting the above equations into this one, we get:

$$\frac{2e}{d} + \frac{2e}{k} > e$$

Dividing both sides by $2e$:

$$\frac{1}{d} + \frac{1}{k} > \frac{1}{2}$$

If we analyze this equation, we discover that d and k can't both be larger than 3. If they were both 4 or above, the left side of the equation would be $\leq \frac{1}{2}$. Since we know that d and k are both ≥ 3, this implies that one of the two is actually equal to three and the other is some integer that is at least 3.

Suppose we set d to be 3. Then the equation becomes $\frac{1}{3} + \frac{1}{k} > \frac{1}{2}$. So $\frac{1}{k} > \frac{1}{6}$, which means that k can't be any larger than 5. Similarly, if k is 3, then d can't be any larger than 5.

This leaves us only five possibilities for the degrees d and k: $(3, 3)$, $(3, 4)$, $(3, 5)$, $(4, 3)$, and $(5, 3)$. Each of these corresponds to one of the Platonic solids.